Ascertainment on the integral solutions of the Sextic Diophantine Equation

$$m^4 - n^4 = 4(u^2+1)(r-s)t^3$$

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Abstract:

The Sextic Diophantine Equation having five unknowns given by $m^4 - n^4 = 4(u^2+1)(r-s)t^3$ is dissected for its non-zero distinct integer solutions. A few fascinating relations among the solutions and the numbers like Triangular number, Pyramidal number, Pronic number, Stella octangula number and Gnomonic numbers are also presented.

Keywords: Sextic Diophantine Equation, integral solutions.

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Introduction:

While individual equations present a kind of puzzle and are thought-about throughout history, the formulation of general theories of Diophantine equations was an accomplishment of the twentieth century. [1-3] gives detailed and clear study on diophantine Equations. For various ideas on problem solving techniques of Diophantine and Exponential Diophantine equations of assorted orders, [4-16] have been referred.

This paper issues with the matter of determining non-trivial integral solutions of the Sextic diophantine equation with five unknowns given by $m^4 - n^4 = 4(u^2+1)(r-s)t^3$. A few attention-grabbing relations among the solutions and the numbers like Triangular number, Pyramidal number, Pronic number, Stella octangula number and Gnomonic numbers are also presented.

NOTATIONS:

$T_{m,n}$ = Triangular Number of rank n.

P_m^n = Pyramidal Number of rank n.

$P(n)$ = Pronic Number of rank n.

Gno_n = Gnomonic Number of rank n.

SO_n = Stella octangula Number of rank n.

Method of Analysis:

The considered Sextic diophantine equation with five unknowns is

$$m^4 - n^4 = 4(u^2 + 1)(r - s)t^3 \quad (1)$$

Introducing the linear transformations,

$$\left.\begin{aligned} m &= p + q \\ n &= p - q \\ r &= 1 + pq \\ s &= 1 - pq \end{aligned}\right\} \quad (2)$$

in (1) and simplifying we get

$$p^2 + q^2 = (u^2 + 1)t^3 \quad (3)$$

Solving the above equation, we obtain an infinite number of integer solutions under various patterns.

Pattern-I:

Let $t = j^2 + k^2$ (4)

Using (4) in (3) and using the method of factorization, define

$$(p + iq) = (u + i)(j + ik)^3$$

Equating real and imaginary parts we get

$$p = uj^3 - 3ujk^2 + k^3 - 3j^2k$$
$$q = -uk^3 + 3uj^2k + j^3 - 3jk^2$$

$$\alpha(j,k) = j^3 - 3jk^2$$
$$\beta(j,k) = k^3 - 3j^2k$$

Therefore p and q becomes

$$p = u\alpha(j,k) + \beta(j,k)$$
$$q = \alpha(j,k) - u\beta(j,k)$$

Hence the nonzero integral solutions to (1) for any u are,

$$m = j^3u - 3jk^2u + k^3u - 3kj^2u + j^3 - 3jk^2 + k^3 - 3j^2k$$
$$n = 3jk^2 - 3j^2k + j^3u + k^3u - j^3 + k^3 - 3jk^2u - 3j^2ku$$

$r = j^6u + 3j^5ku^2 - 3j^5k - 15j^4k^2u - 10j^3k^3u^2 + 10j^3k^3 + 15j^2k^4u + 3jk^5u^2 - 3jk^5 - k^6u + 1$

$s = 3j^5k - 3j^5ku^2 - j^6u + 15j^4k^2u + 10j^3k^3u^2 - 10j^3k^3 - 15j^2k^4u - 3jk^5u^2 + 3jk^5 + k^6u + 1$

Some numerical examples are listed in the table below in table1.

TABLE 1									
u	j	k	m	n	r	s	t	L.H.S	R.H.S
1	5	4	-230	-472	-42470	42472	41	-46834300656	-46834300656
2	7	2	1063	-599	192793	-192791	53	1148091783360	1148091783360
4	2	2	-32	-128	-3839	3841	8	-267386880	-267386880
-1	5	6	-468	830	-117468	117470	61	-426611697424	-426611697424
0	2	3	-55	37	415	-413	13	7276464	7276464

OBSERVATIONS:

1. $\frac{m(u,0) + n(u,0) - t(u,0)}{24} = 4DF_n$
2. $m(u,-1) + (u+1) = 24PT_n - 6P_n^5 - 8\Pr_n$
3. $n(0,b) + b^3(1-u) = 0$
4. $3[r(u,0) + s(u,0)]$ and $3t(u,u)$ are nasty numbers.

Pattern-II:

Rewriting (3) as,

$$p^2 + q^2 = 1*(u^2+1)t^3 \qquad (5)$$

Write 1 as

$$1 = \frac{3^2+4^2}{5^2} \text{ and let } t = j^2 + k^2 \qquad (6)$$

Using (6) in (5) and proceeding as in pattern-I we get,

$$p = \frac{u(3j^3 + 4k^3 - 9jk^2 - 12j^2k) - (4j^3 - 3k^3 - 12jk^2 - 9j^2k)}{5}$$

$$q = \frac{u(4j^3 - 3k^3 - 12jk^2 + 9j^2k) + (3j^3 + 4k^3 - 9jk^2 - 12j^2k)}{5}$$

Since the objective is to find the integer solution, taking j = 5J and k = 5K and we get,

$$p = u\alpha(J,K) - \beta(J,K)$$

$$q = u\beta(J,K) + \alpha(J,K)$$

where

$$\alpha(J,K) = 75J^3 + 100K^3 - 225JK^2 - 300J^2K$$

$$\beta(J,K) = 100J^3 - 75K^3 - 300JK^2 + 225J^2K$$

Hence the nonzero integral solutions to (1) are,

$$m = 150uJ^3 + 200uK^3 - 450uJK^2 - 600uKJ^2$$

$$n = 1183uJ^3 + 2873uK^3 - 3549uJK^2 - 8619uJ^2K - 2873J^3 + 1183K^3 + 8619JK^2 - 3549J^2K$$

$$r = 1 + 150000K^3J^3 - 45000K^5J - 45000KJ^5 + 150000u^2J^3K^3 - 45000u^2J^5K - 45000u^2JK^5 - 10000J^6 + 56250u^2J^4K^2 - 9375u^2K^4J^2 - 5625K^6 + 5625u^2J^6 + 10000u^2K^6 + 9375J^4K^2 - 56250K^4J^2$$

$$s = 1 - 150000K^3J^3 + 45000K^5J + 45000KJ^5 - 150000u^2J^3K^3 + 45000u^2J^5K + 45000u^2JK^5 + 10000J^6 - 56250u^2J^4K^2 + 9375u^2K^4J^2 + 5625K^6 - 5625u^2J^6 - 10000u^2K^6 - 9375J^4K^2 + 56250K^4J^2$$

$$t = (5J)^2 + (5K)^2$$

Some numerical examples are listed below in table 2.

TABLE 2

u	J	K	m	n	r	s	t	LHS=RHS
1	2	3	-8700	7850	3516876	-3516874	325	1931643593750000
2	3	4	-47925	12025	538051251	-538051249	625	5254406738281250000
-2	4	3	1875	49375	-608593749	608593751	625	-5943298339843750000

OBSERVATIONS:

1. $m(u,0) - t(u,u) = 600(4DF_n)$
2. $5000SO_u + u[1 - r(0,1) + 625] = 0$
3. $n(a,a) + u.t(u,0) =$ a cubic number
4. $u.s(\sqrt{u},0) + 5625u^6 - u =$ a biquadratic integer
5. $m(0,b) + n(b,0) = 200(\mathrm{Pr}_n^2 - 6P_n^5 + T_{6,n} + 1)$

Pattern-III:

1 in (6) can also be written as

$$1 = \frac{12^2 + 5^2}{13^2} \tag{7}$$

Using (7) in (5) and proceeding as in pattern-II we get,

$$p = \frac{u(12j^3 + 5k^3 - 15j^2k - 36jk^2) - (5j^3 - 12k^3 - 15jk^2 + 36j^2k)}{13}$$

$$q = \frac{u(5j^3 - 12k^3 - 15jk^2 + 36j^2k) + (12j^3 + 5k^3 - 36jk^2 - 15j^2k)}{13}$$

Since the objective is to find the integer solution, taking j = 13J and k = 13K and we get,

$$p = u\alpha(J,K) - \beta(J,K)$$

$$q = u\beta(J,K) + \alpha(J,K)$$

where

$\alpha(J,K) = 2028J^3 + 845K^3 - 6084JK^2 - 2535J^2K$

$\beta(J,K) = 845J^3 - 2028K^3 - 2535JK^2 + 6084J^2K$

Hence the nonzero integral solutions to (1) are,

$m = 2873uJ^3 - 1183uK^3 - 8619uJK^2 + 3549uJ^2K + 1183J^3 + 2873K^3 - 3549JK^2 - 8619J^2K$

$n = 1183uJ^3 + 2873uK^3 - 3549uJK^2 - 8619uJ^2K - 2873J^3 + 1183K^3 + 8619JK^2 - 3549J^2K$

$$\begin{aligned} r = {} & 1 + 1713660u^2J^6 + 3398759uJ^6 - 1713660u^2K^6 - 3398759uK^6 + 33987590J^3K^3 \\ & - 33987590u^2J^3K^3 - 25704900u^2J^4K^2 + 10196277u^2J^5K + 68546400uJ^3K^3 - 50981385uJ^4K^2 \\ & - 20563920uJ^5K + 10196277u^2K^5J + 25704900u^2K^4J^2 - 20563920uK^5J + 50981385uK^4J^2 \\ & - 1713660J^6 + 1713660K^6 + 25704900J^4K^2 - 10196277J^5K - 10196277K^5J - 25704900K^4J^2 \end{aligned}$$

$$\begin{aligned} s = {} & 1 - 1713660u^2J^6 - 3398759uJ^6 + 1713660u^2K^6 + 3398759uK^6 - 33987590J^3K^3 \\ & + 33987590u^2J^3K^3 + 25704900u^2J^4K^2 - 10196277u^2J^5K - 68546400uJ^3K^3 + 50981385uJ^4K^2 \\ & + 20563920uJ^5K + 10196277u^2K^5J - 25704900u^2K^4J^2 + 20563920uK^5J - 50981385uK^4J^2 \\ & + 1713660J^6 - 1713660K^6 - 25704900J^4K^2 + 10196277J^5K + 10196277K^5J + 25704900K^4J^2 \end{aligned}$$

$t = (13J)^2 + (13K)^2$

Some numerical examples are listed below in table 3.

TABLE 3

u	J	K	m	n	r	s	t	LHS=RHS
2	3	4	-833001	-245557	158398106439	-158398106437	4225	477848378586185000000000
1	4	3	-376194	-400192	-4657927806	4657927808	4225	-56207319650444700000000
-2	2	3	162747	282061	-13267955427	13267955429	2197	-56280010006887200000000

OBSERVATIONS:

1. $m(u,0) - n(0,u) + 378u^3 + 815u^4 =$ sum of cubic and biquadratic integer
2. $m(u,0), n(0,u) = 0$
3. $t(u,u) + 23\,\mathrm{Pr}_u + 23u = a$ square number
4. $t(u,u) + 23\,\mathrm{Pr}_u + 23u \equiv 0 (\mathrm{mod}\, 19)$
5. $m(0,b) + n(b,0) = 200(\mathrm{Pr}_n^2 - 6P_n^5 + T_{6,n} + 1)$

Pattern-IV:

Write 1 as

$$1 = \frac{24^2 + 7^2}{25^2} \qquad (8)$$

Using (8) in (5) and proceeding as in pattern-II we get,

$$p=\frac{u(24j^3+7k^3-21j^2k-72jk^2)-(7j^3-24k^3-21jk^2+72j^2k)}{25}$$

$$q=\frac{u(7j^3-24k^3-21jk^2+72j^2k)+(24j^3+7k^3-72jk^2-21j^2k)}{25}$$

Since the objective is to find the integer solution, taking j = 25J and k = 25K we get,

$$p=u\alpha(J,K)-\beta(J,K)$$

$$q=u\beta(J,K)+\alpha(J,K)$$

where

$$\alpha(J,K)=15000J^3+4375K^3-13125J^2K-45000JK^2$$

$$\beta(J,K)=4375J^3-15000K^3+45000J^2K-13125JK^2$$

Hence the nonzero integral solutions to (1) are,

$$m=19375uJ^3-10625uK^3+31875uJ^2K-58125uJK^2+10625J^3+19375K^3-58125J^2K-31875JK^2$$

$$n=10625uJ^3+19375uK^3-58125uJ^2K-31875uJK^2-19375J^3+10625K^3-31875J^2K+58125JK^2$$

$$\begin{aligned}r=1&+65625000u^2J^6+205859375uJ^6-65625000u^2K^6-205859375uK^6+2058593750J^3K^3\\&-2058593750u^2J^3K^3-984375000u^2J^4K^2+617578125u^2J^5K+2625000000uJ^3K^3\\&-3087890625uJ^4K^2-787500000uJ^5K+617578125u^2K^5J+984375000u^2K^4J^2\\&-787500000uK^5J+3087890625uK^4J^2-65625000J^6+65625000K^6+984375000J^4K^2\\&-617578125J^5K-617578125K^5J-984375000K^4J^2\end{aligned}$$

$$\begin{aligned}s=1&-65625000u^2J^6-205859375uJ^6+65625000u^2K^6+205859375uK^6-2058593750J^3K^3\\&+2058593750u^2J^3K^3+984375000u^2J^4K^2-617578125u^2J^5K-2625000000uJ^3K^3\\&+3087890625uJ^4K^2+787500000uJ^5K-617578125u^2K^5J-984375000u^2K^4J^2\\&+787500000uK^5J-3087890625uK^4J^2+65625000J^6-65625000K^6-984375000J^4K^2\\&+617578125J^5K+617578125K^5J+984375000K^4J^2\end{aligned}$$

$$t=(25J)^2+(25K)^2$$

Some numerical examples are listed below in table 4.

TABLE 4

u	J	K	m	n	r	s	t	LHS=RHS
2	3	4	-5694375	-2391875	6676210156251	-6676210156249	15625	1018708825111390000000000000
-1	4	3	-3125000	2343750	1068115234376	-1068115234374	15625	65192580223083500000000000
1	2	3	-1458750	132500	527598828126	-527598828124	8125	4527869630432130000000000
3	5	4	-4956250	-17662500	-71849873046874	71849873046876	25625	-96718068364524800000000000000

OBSERVATIONS:

1. $t(u,u)+34u^2$ is a quadratic integer
2. $t(u,u)+36u^2$ is a nasty number
3. $t(u,u)+2T_{38,n}+17Gno_n-486u^2\equiv 0 \pmod{17}$

Conclusion:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the considered Sextic diophantine equation having five under different patterns. One can also search for other patterns of solutions for the above equation.

References:

[1] L.E.Dickson, History of the Theory of Numbers, Chelsea Publishing Company, New York, Vol II, 1952.

[2] Niven, Ivan, Zuckerman, S.Herbert and Montgomery, L.Hugh, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc, New York, 2004.

[3] L.J. Mordell, Diophantine equations, Academic Press, New York, 1969.

[4] Gopalan MA, Janaki G, Integral solutions of $(x^2-y^2)(3x^2+3y^2-2xy)=2(z^2-w^2)p^3$, Impact J.Sci Tech, Vol 4, PP.97-102, 2010.

[5] M.A.Gopalan, S.Vidhyalakshmi, D.Maheswari, Observations on the Quintic Equation with five unknowns $x^4-y^4=37(z^2-w^2)p^3$, International Journal of Applied Research , Vol 1 ,Issue 3,PP. 78-81,2015.

[6] Gopalan MA, Vidhyalakshmi S, Kavitha A. On the quintic equation with five unknowns, $2(x-y)(x^3+y^3)=19(z^2-w^2)p^3$ International Journal of Engineering Research 2013, Vol 1, Issue 2 PP.279-282.

[7] Vidhyalakshmi S, Mallika S, Gopalan MA. Observations on the non-homogeneous quintic equation with five unknowns $x^4-y^4=2(k^2+s^2)(z^2-w^2)p^3$,International Journal of Innovative Research in Science, Engineering and Technology , Vol 2, Issue 4, PP.1216-1221, 2013

[8] Gopalan MA, Vidhyalakshmi S, Premalatha E and Manjula S ,The Non-Homogeneous Quintic Equation With Five Unknowns $x^4-y^4+2(x^2+y^2)(x-y)^2=14(z^2-w^2)p^3$ International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online) Vol. 5 ,Issue 1, January-March, PP. 64-69,2015.

[9] Gopalan MA., Vidhyalakshmi S , Thiruniraiselvi N And Malathi R, The Non- homogeneous Quintic Equation With Five Unknowns $x^4-y^4=65(z^2-w^2)p^3$ International Journal Of Physics And Mathematical Sciences) , Vol 5, Issue 1, January-March, PP.70-77,2015.

[10] G Janaki, P Saranya, On the Ternary Quadratic Diophantine Equation, $5(x^2+y^2)-6xy=4z^2$, Imperial Journal of Interdisciplinary Research, Vol 2, Issue 3, PP: 396-397, 2016.

[11] G Janaki , P Saranya, On the Ternary cubic Diophantine Equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$, Imperial Journal of Interdisciplinary Research, Vol 5, Issue 3, PP: 227-229, 2016.

[12] P Saranya, G Janaki, On the Quintic Non-Homogeneous Diophantine Equation $x^4-y^4=40(z^2-w^2)p^3$, International Journal of Engineering Science and Computing, Vol 7, Issue 2, PP: 4685-4687, 2016.

[13] P Saranya, G Janaki, On the Exponential Diophantine Equation, $36^x+3^y=z^2$, International Research Journal of Engineering and Technology, Vol 4, Issue 11, PP:1042-1044, 2017.

[14] P Saranya, G Janaki, Explication of the Transcendental Equation $p+\sqrt{p^3+q^3-pq}+\sqrt[3]{r^2+s^2}=(m^2+1)t^3$, Adalya Journal, Vol 8, Issue 9, PP:70-73, 2019.

[15] P Saranya, G Janaki, Ascertainment On The Exponential Equation $p^{3c}-a(p^{2c}-p^b)=p^{b+c}$, Compliance Engineering Journal, Vol 10, Issue 8, PP:225-229, 2019.

[16] Manjusomanath, G. Sangeetha, M.A. Gopalan, On the Sextic Diophantine Equation with Three Unknowns $x^2-xy+y^2=7z^6$, Scholars Journal of Engineering and Technology (SJET), Vol 2 , Issue 3A , PP: 377-379, 2014.